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An entanglement measure based on two-order minors

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Abstract

In this paper, a new entanglement measure called EMM (the entanglement measure based on minors) has been constructed by the convex roof method and proved to be a good entanglement measure according to the axiomatic point of view. Computation of EMM can be finished directly by the two-order minors of a coefficient matrix, instead of the eigenvalues of the density operator required by most of the other entanglement measures, so it is very fast and very easy. On the other hand, EMM($|\psi\rangle$) is related to the modified Bures distance between $|\psi\rangle$ and the closest separable state, so it has geometry meaning. We also investigate the relations between EMM and the entanglement of formation, negativity and logarithmic negativity, and discover that EMM is always smaller than or equal to them. EMM is equivalent to concurrence. However, their definitions and methods of proof are completely different. EMM will improve the efficiency of searching for maximally entangled multipartite states.

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1. Introduction

Quantum entanglement is one of the most fascinating features in quantum theory, which makes it central to quantum information theory [1]. Quantum entanglement plays a key role in many application fields such as quantum teleportation [2], quantum dense coding [3], quantum key distribution [4], quantum secret sharing [5], etc.

An n-partite state ρ acting on Hilbert space $H_1 \bigotimes H_2 \bigotimes \cdots \bigotimes H_n$ being separable means that there exists a decomposable ensemble $\left\{p_i, \bigotimes_{j=1}^{n} \rho_i^j\right\}\left(p_i > 0, \sum_i p_i = 1\right)$ to realize ρ, that is to say, $\rho = \sum_i p_i\left(\otimes_{j=1}^{n} \rho_i^j\right)$. Particularly, an n-partite pure state,

$|\phi\rangle \in H_1 \bigotimes H_2 \bigotimes \cdots \bigotimes H_n$, is said to be separable, if it can be written in the form $|\phi^n\rangle = \bigotimes_{i=1}^n |\psi_i\rangle$, where $|\psi_i\rangle \in H_i (i = 1, 2, \ldots, n)$. A multipartite state ρ is said to be entangled if it is not separable [6].

One foundational task characterizing entanglement is quantifying the degree of entanglement [7, 8]. The first two entanglement measures proposed are the entanglement of distillation [7, 9] and the entanglement cost [7, 10]. These measures have direct physical significance. The entanglement cost of ρ quantifies the maximal possible rate at which one can convert input (the maximally entangled states of two-qubit) into output states that approximate ρ. The entanglement of distillation of ρ qualifies at what rate we may obtain maximally entangled states from plenty of ρ.

The measures such as the entanglement of distillation and the entanglement cost are built to describe the entanglement in terms of some tasks. Thus they arise from the optimization of some protocols performed on quantum states. Therefore, one of the main disadvantages of them is difficulty in computation of their values. For this reason, many other entanglement measures have been constructed. The most important method to construct an entanglement measure is based on the axiomatic point of view, by allowing any function of states to be a measure, provided it satisfies some postulates [7, 8, 11]. Vedral [8] introduced the idea of axiomatic definition of entanglement measures and proposed that an entanglement measure is any function that satisfies the postulates below:

(i) Entanglement vanishes on separable states and reaches a maximal value on generalized Bell States.
(ii) Entanglement remains invariant under local unitary transformation.
(iii) Entanglement cannot increase under local operation and classical communication, i.e., LOCC.

Based on these postulates, many entanglement measures have been proposed over recent years, such as the relative entropy of entanglement [8, 11], the entanglement of formation based on Von Neumman entropy [10], the entanglement of formation based on Renyi entropy [12], the squashed entanglement [13], negativity [14, 15], logarithmic negativity [15, 16], norm-based entanglement [17], witnessed entanglement [18], the max-relative entropy of entanglement [19], the robustness of entanglement [20], the Schmidt number [21] and the entanglement measure based on covariance [22]. A family of entanglement measures built by means of polynomials of the Schmidt coefficients was introduced by Sinolecka *et al* [23] and developed by Gour [24].

Some other postulates, e.g., normalization, asymptotic continuity, convexity and additivity, have been extended. However, it is accepted to all that the three postulates above are essential.

The convex roof method [25] can be used to construct entanglement measures: one starts by imposing a measure on pure states, and then extends it to mixed ones by the convex roof,

$$E(\rho) = \inf \sum_i p_i E(|\psi_i\rangle), \qquad \sum_i p_i = 1, \qquad p_i \geqslant 0, \tag{1}$$

where the infimum is taken over all ensembles $\{p_i, |\psi_i\rangle\}$ for which $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. The first entanglement built in this way was the entanglement of formation introduced by Bennett [10], where $E(|\psi_i\rangle)$ is the Von Neumann entropy of the reduced density matrix of $|\psi_i\rangle\langle\psi_i|$. Vidal [12] generalized the entanglement of formation based on Von Neumman entropy to the entanglement of formation based on Renyi entropy. Another important entangle measure created by the convex roof method is concurrence due to Hill and Wootters [26], which is defined with the help of the qubit spin-flip operator for a pure state of two-qubit, and is extended to mixed states by the convex roof. Moreover, Wootters [27] showed that, for an arbitrary two-qubit mixed state ρ, there exist explicit formulae of concurrence, and entanglement of formation of ρ is the monotonously increasing determinate function of concurrence of ρ. Rungta *et al* [28] generalized Wootters's concurrence to a bipartite quantum system of arbitrary dimensions with the help of a 'universal inverter' imitating the spin-flip operator and drew an explicit formula

$$C(|\Psi_{AB}\rangle) = \sqrt{2\left[1 - \mathrm{tr}\left(\rho_A^2\right)\right]}, \tag{2}$$

where ρ_A represents the marginal density operator of the subsystem A.

Vidal [12] and Horodecki [29] have proved in different methods that if a measure satisfies Vedral postulates on pure states, then its convex roof extension is also an entanglement measure on mixed states. So, if a function $E(|\psi\rangle)$ satisfies Vedral postulates on pure states, then it is a sound entanglement measure for the general state ρ (including pure states and mixed states).

So far, all the entanglement measurements are difficult to be calculated. Some entanglement measurements, such as the entanglement cost, the distilled entanglement, the squashed entanglement and witnessed entanglement, etc, are operational motivated, and generally involve an optimization over high-dimensional spaces which makes their evaluation exceedingly difficult. Almost all of the other entanglement measurements are essentially based on the eigenvalues of density operators, such as the entanglement of formation, the relative entropy of entanglement, the max-relative entropy of entanglement, negativity, concurrence and a family of polynomials of the Schmidt coefficients developed by Sinolecka *et al* [23] and Gour [24]. It is known to all that the computation of the eigenvalues of density operators acting on high-dimensional Hilbert space is very complicated and takes much time. Multipartite maximally entangled pure states play a key role in various quantum information processing tasks [30–34], but we know little about them. Recently, several research groups [35, 36] began to search for the maximally entangled multipartite quantum pure states based on the entanglement measures of negativity and Von Neumman entropy, etc, by a numerical optimization procedure, but the high computation complexity of the entanglement measurements based on the eigenvalues of density operators tampers with badly the efficiency of accomplishing the task. At present, the search task is fulfilled only in the state space of no more than seven qubits. So, searching for an entanglement measurement with low complexity of computation is of great interest, which motivates our research in this paper.

In another of our papers [37], we present a simple separability criterion based on two-order minors of the coefficient matrices for two-partite pure states. Just as the PPT separability criterion [38, 39] induces an entanglement measure, negativity, we guess that this separability criterion implies an entanglement measure, which can be defined as

$$E(|\phi\rangle) = 4 \sum_{i\neq i', j\neq j'} |a_{ij}a_{i'j'} - a_{ij'}a_{i'j}|^2, \tag{3}$$

for a pure state $|\phi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} a_{i,j}|i, j\rangle$ and defined as equation (1) for a mixed state ρ. Here, $a_{ij}a_{i'j'} - a_{ij'}a_{i'j}$ represents a two-order minor of the coefficient matrix $(a_{ij})_{d_1\times d_2}$. For convenience, we call the entanglement measure based on minors EMM. EMM can be computed quickly and easily since it can be finished directly by the two-order minors of the coefficient matrices, but not by the Schmidt coefficients.

In this paper, we prove this guess, i.e., equation (3) is suitable for an entanglement measure. In section 2, we present some lemmas which are required by the following section, and the proofs of lemmas 2–4 are moved to appendices for clarity. Our main result that

EMM is a good entangle measure is placed in section 3. In section 4, we give a geometry interpretation about EMM. Finally, we conclude with a summary in section 5.

2. Preliminaries

First, let us give this separability criterion based on two-order minors, and the proof is referred to paper [37].

Lemma 1 (Separability criterion based on minor). *Suppose a two-partite qudit pure state* $|\phi\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} a_{i,j}|i, j\rangle$ *and its corresponding coefficient matrix is* $\Phi = (a_{ij})_{d_1\times d_2}$. *Then,* $|\phi\rangle$ *is separable if and only if all the two-order minors of* Φ *are vanishing.*

Now, we will present some properties of the two-order minors of the matrix, which are stated by lemmas 2–5. The proofs of them are placed in appendices A–D, respectively.

Lemma 2. *Suppose that* $\overrightarrow{a} = (a_1, a_2, \ldots, a_n)^T$ *and* $\overrightarrow{b} = (b_1, b_2, \ldots, b_n)^T$ *are orthogonal unit vectors in a complex field C, i.e.,* $|\overrightarrow{a}| = 1, |\overrightarrow{b}| = 1, (\overrightarrow{a}, \overrightarrow{b}) = 0$. *Then*

$$\sum_{i<j} |a_i b_j - a_j b_i|^2 = 1. \tag{4}$$

That is to say, the sum of the square of the module of the two-order minors of the unit orthogonal $n \times 2$ *matrix* $(\overrightarrow{a}, \overrightarrow{b})$ *is equal to 1.*

The result above can be generalized to unnormalized orthogonal $n \times 2$ matrices, which is stated by corollary 1.

Corollary 1. *Suppose that* $\overrightarrow{a} = (a_1, a_2, \ldots, a_n)^T$ *and* $\overrightarrow{b} = (b_1, b_2, \ldots, b_n)^T$ *are orthogonal vectors in a complex field C and* $|\overrightarrow{a}| = \lambda, |\overrightarrow{b}| = \mu$. *Then,*

$$\sum_{i<j} |a_i b_j - a_j b_i|^2 = \lambda^2\mu^2. \tag{5}$$

Lemma 3. *Suppose that any two rows and any two columns of* $A = (a_{ij})_{d_1\times d_2}$ *are orthogonal, respectively.* $U = (u_{ij})_{d_1\times d_1}$ *and* $V = (v_{ij})_{d_2\times d_2}$ *are unitary matrices. Let* $B = UA, C = AV, A(kr, js) = a_{kr}a_{js} - a_{ks}a_{jr}, B(mr, ns) = b_{mr}b_{ns} - b_{ms}b_{nr}, C(mr, ns) = c_{mr}c_{ns} - c_{ms}c_{nr}$. *Then,*

(i) $\sum_{m<n,r<s} |B(mr, ns)|^2 = \sum_{k<j,r<s} |A(kr, js)|^2$,
(ii) $\sum_{m<n,r<s} |C(mr, ns)|^2 = \sum_{k<j,r<s} |A(kr, js)|^2$.

Lemma 3 means that the sum of the square of the module of the two-order minors of the orthogonal matrix remains invariant under unitary transformation.

Lemma 4. *For a two-partite qudit pure state* $|\phi\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} a_{i,j}|i, j\rangle$. *Suppose its Schmidt decomposition is* $|\phi\rangle = \sum_{i=1}^{d}\sqrt{\lambda_i}|\mu_i\rangle|\nu_i\rangle$. *Here, d is the Schmidt number of state* $|\phi\rangle$, λ_i *are nonnegative real numbers satisfying* $\sum_{i=1}^{d}\lambda_i = 1$, $\{|\mu_i\rangle \mid 1 \leqslant i \leqslant d\}$, *and* $\{|\nu_i\rangle \mid 1 \leqslant i \leqslant d\}$ *are orthogonal quantum states in the Hilbert spaces* H_1 *and* H_2, *respectively. Denote* $\sum_{i<i',j<j'} |a_{ij}a_{i'j'} - a_{ij'}a_{i'j}|^2$ *by Minors*(Φ). *Then,*

$$Minors(\Phi) = \frac{1}{2}\sum_{i=1}^{d}\lambda_i(1 - \lambda_i). \tag{6}$$

Lemma 4 relates the quantity Minors(Φ) with a polynomial function of Schmidt coefficients.

Lemma 5. *Suppose* $\overrightarrow{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ *be a d-dimensional real vector, where* λ_i*'s are nonnegative real numbers satisfying* $\sum_{i=1}^{d} \lambda_i = 1$. *Denote* $\frac{1}{2}\sum_{i=1}^{d} \lambda_i(1-\lambda_i)$ *by* $f(\overrightarrow{\lambda})$. *Then* $f(\overrightarrow{\lambda}) \leqslant \frac{1}{2}\left(1-\frac{1}{d}\right)$*, and the equality is true if and only if* $\forall\, i, \lambda_i = \frac{1}{d}$.

By lemma 5, we have Minors(Φ) $\leqslant \frac{1}{2}\left(1-\frac{1}{d}\right)$.

Lemma 6. *Suppose that* $|\phi\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} a_{i,j}|i, j\rangle$ *is an entangled state on* $(d_1 \times d_2)$*-dimensional Hilbert space* $H_A \otimes H_B$*, and its Schmidt form is*

$$|\phi\rangle = \sum_{j}^{d} \sqrt{\lambda_j}|u_j\rangle|v_j\rangle, \qquad d = min(d_1, d_2). \tag{7}$$

Here, $\{|u_j\rangle | j = 1, \ldots, d\}$ *and* $\{|v_j\rangle | j = 1, \ldots, d\}$ *are orthogonal states set chosen so that the Schmidt coefficients* λ_j *are in increasing order, i.e.*

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_d. \tag{8}$$

Then an ensemble $\{p_k, |\phi_k\rangle | k = 1, \ldots, N\}$ *can be produced from* $|\phi\rangle$ *by a suitable LOCC if and only if the majorization inequalities,*

$$\sum_{k}^{N} p_k E_n(|\phi_k\rangle) \leqslant E_n(|\phi\rangle), \tag{9}$$

hold for $1 \leqslant n \leqslant d$*, where*

$$E_n(|\phi\rangle) = \sum_{j=1}^{n} \lambda_j, \tag{10}$$

and similarly for $|\phi_k\rangle$.

Proof. This lemma is due to Jonathan and Plenio, and please refer to [40] for the proof. □

Lemma 6 presents the sufficient and necessary conditions based on Schmidt coefficients that a pure state $|\phi\rangle$ can be transformed into an ensemble $\{p_k, |\phi_k\rangle | k = 1, \ldots, N\}$ by a suitable LOCC.

Lemma 7. *Let* $\vec{x} = (x_1, x_2, \ldots, x_d)$ *and* $\vec{y} = (y_1, y_2, \ldots, y_d)$ *be the d-dimensional real vectors with* x_i *and* y_i *being arranged in increasing order, respectively. Suppose that* $f : R \to R$ *is any convex function,* $F(\vec{x}) = \sum_{i=1}^{d} f(x_i)$. *Then :* $F(\vec{x}) \leqslant F(\vec{y})$ *if and only if* $\vec{x} \prec \vec{y}$. *Here,* $\vec{x} \prec \vec{y}$ *means that* $\vec{x}$ *is majorized by* $\vec{(y)}$*, i.e.,* $\sum_{i=1}^{n} x_i \geqslant \sum_{i=1}^{n} y_i, 1 \leqslant n \leqslant d$.

Proof. Please refer to Alberti's book [41] for the proof. □

Lemma 7 shows that $F(\vec{x}) = \sum_{i=1}^{d} f(x_i)$ is Schur-convex if and only if $f(x)$ is a convex function.

3. Entanglement measure based on two-order minors

In this section, we will prove that EMM is a good entanglement measure for qudit pure states. For this aim, we only need to prove the theorem below.

Theorem 1. *For a two-partite qudit pure state* $|\phi\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} a_{i,j}|i, j\rangle$, *Minors*$(\Phi) = \sum_{i\neq i', j\neq j'} |a_{ij}a_{i'j'} - a_{ij'}a_{i'j}|^2$ *satisfies the following properties:*

(i) Minors$(\Phi) = 0$ *if and only if* $|\phi\rangle$ *is separable.*
(ii) Minors(Φ) *reaches maximum if and only if* $|\phi\rangle$ *is a maximal entanglement state.*
(iii) Minors(Φ) *remains invariant under local unitary transformation.*
(iv) Minors(Φ) *is not increasing under LOCC.*

Proof.

(i) (If) Suppose that $|\phi\rangle$ is separable. Then, by lemma 1, all the two-order minors of Φ are zero, i.e., $a_{ij}a_{i'j'} - a_{ij'}a_{i'j} = 0$ for $1 \leqslant i, i' \leqslant d_1, 1 \leqslant j, j' \leqslant d_2$. It follows that $\sum_{i\neq i', j\neq j'} |a_{ij}a_{i'j'} - a_{ij'}a_{i'j}|^2 = 0$, i.e., Minors$(\Phi) = 0$. (Only if) Suppose Minors$(\Phi) = 0$. Then, $a_{ij}a_{i'j'} - a_{ij'}a_{i'j} = 0\ \forall 1 \leqslant i, i' \leqslant d_1, 1 \leqslant j, j' \leqslant d_2$. It follows that all the two-order minors of Φ are zero. By lemma 1, we know that $|\phi\rangle$ is separable.

(ii) It can be immediately obtained by lemmas 4 and 5.

(iii) Suppose that U and V are unitary transformation acting on the first system and the second system, respectively. Then the state $(U \bigotimes V)|\phi\rangle$ corresponds to the coefficient matrix $U\Phi V$, denoted by Φ'. The singular-value decomposition theorem says that there exist unitary matrices U_1, V_1 such that $\Phi = U_1\Sigma V_1$. Here,

$$\Sigma = \begin{pmatrix} \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_d}) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\sqrt{\lambda_i}$'s are the singular value of Φ, i.e., the Schmidt coefficients of the state $|\phi\rangle$. So,

$$\Phi' = UU_1\Sigma V_1 V. \tag{11}$$

Obviously, matrices UU_1, VV_1 are unitary matrices too. By lemma 4, we have

$$\mathrm{Minors}(\Phi') = \frac{1}{2}\sum_{i=1}^{d}\lambda_i(1-\lambda_i) = \mathrm{Minors}(\Phi). \tag{12}$$

(iv) The key point of the proof is to synthesize the result of lemmas 4, 6 and 7. Suppose that the ensemble produced from $|\phi\rangle$ by LOCC is $\{p_k, |\phi_k\rangle | k = 1, \ldots, N\}$, and the Schmidt form of $|\phi\rangle$ is

$$|\phi\rangle = \sum_{j=1}^{d}\sqrt{\lambda_j}|u_j\rangle|v_j\rangle. \tag{13}$$

Here, $d = \min(d_1, d_2)$, $\{|u_j\rangle | j = 1, \ldots, d\}$ and $\{|v_j\rangle | j = 1, \ldots, d\}$ are orthogonal sets chosen so that the Schmidt coefficients λ_j's are in increasing order, i.e.

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_d. \tag{14}$$

Similarly, suppose that the Schmidt form of $|\phi_k\rangle$ is

$$|\phi_k\rangle = \sum_{j}^{d}\sqrt{\mu_{kj}}|c_{kj}\rangle|d_{kj}\rangle. \tag{15}$$

Here, $\{|c_{kj}\rangle | j = 1, \ldots, d\}$ and $\{|d_{kj}\rangle | j = 1, \ldots, d\}$ are orthogonal sets chosen so that the Schmidt coefficients μ_{kj} are in increasing order, i.e.

$$0 \leqslant \mu_{k1} \leqslant \mu_{k2} \leqslant \cdots \leqslant \mu_{kd}. \tag{16}$$

By lemma 6, we have the majorization inequalities:

$$\sum_k \sum_{i=1}^{n} p_k \mu_{ki} \leqslant \sum_{i=1}^{n} \lambda_i. \tag{17}$$

That is to say, $(\overrightarrow{\lambda_i}) \prec \left(\overrightarrow{\sum_k p_k u_{ki}}\right)$. By lemma 4, we know that the entanglement measures of $|\phi\rangle$ and $|\phi_k\rangle$ are

$$\mathrm{EMM}(|\phi\rangle) = 4 \times \mathrm{Minors}(\Phi) = 2\sum_{i=1}^{d} \lambda_i (1 - \lambda_i) = 2\left(1 - \sum_{i=1}^{d} \lambda_i^2\right) \tag{18}$$

and

$$\mathrm{EMM}(|\phi_k\rangle) = 4 \times \mathrm{Minors}(\Phi_k) = 2\sum_{i=1}^{d} \mu_{ki}(1 - \mu_{ki}) = 2\left(1 - \sum_{i=1}^{d} \mu_{ki}^2\right), \tag{19}$$

respectively. So, the entanglement measure of the ensemble $\{p_k, |\phi_k\rangle\}$ is

$$\sum_k p_k\, \mathrm{EMM}(|\phi_k\rangle) = \sum_k 4p_k\, \mathrm{Minors}(\Phi_k) = 2\left(1 - \sum_k \sum_{i=1}^{d} p_k \mu_{ki}^2\right). \tag{20}$$

Since the function x^2 is convex, lemma 7 and equation (17) show that

$$2\left(1 - \sum_{i=1}^{d} \left(\sum_k p_k \mu_{ki}\right)^2\right) \leqslant 2\left(1 - \sum_{i=1}^{d} \lambda_i^2\right). \tag{21}$$

By the property of convex functions, we have

$$\left(\sum_k p_k \mu_{ki}\right)^2 \leqslant \sum_k p_k \mu_{ki}^2. \tag{22}$$

It follows that

$$2\left(1 - \sum_k \sum_{i=1}^{d} p_k \mu_{ki}^2\right) \leqslant 2\left(1 - \sum_{i=1}^{d} \left(\sum_k p_k \mu_{ki}\right)^2\right). \tag{23}$$

By combining inequalities (23) and (21), we have

$$2\left(1 - \sum_k \sum_{i=1}^{d} p_k \mu_{ki}^2\right) \leqslant 2\left(1 - \sum_{i=1}^{d} \lambda_i^2\right). \tag{24}$$

That is to say, the entanglement measure of the ensemble $\{p_k, |\phi_k\rangle\}$ is not greater than that of $|\psi\rangle$. □

It is worth pointing out that the result that Minors(ϕ) is not increasing under LOCC can be generalized to that Minors(ϕ) is not increasing under an arbitrary separable operation. This is because the result of lemma 6 can be generalized from LOCC to general separable operation [42].

Theorem 2. *For a qudit pure state* $|\phi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} a_{i,j} |i, j\rangle$, $EMM(|\phi\rangle) \leqslant E_{VN}(|\phi\rangle)$, *where* $EMM(|\phi\rangle) = 4\sum_{i\neq i', j\neq j'} |a_{ij}a_{i'j'} - a_{ij'}a_{i'j}|^2$, $E_{VN}(|\phi\rangle) = -\sum_i p_i \log p_i$ *represent* *the entanglement measure based on minors and the entanglement measure based on Von Neumman entropy, respectively.*

Proof. Let $f(x) = \log_2\left(\frac{1}{\sqrt{x}}\right) + x - 1$. Then it is easy to validate that $f(x)$ is a convex function. Suppose that the Schmidt coefficients of $|\phi\rangle$ are $\{\sqrt{\lambda_j}|1 \leqslant j \leqslant d\}$. Then $p_j = \lambda_j, j = 1, \dots, d$. By the property of convex functions, we have

$$\sum_{i=1}^{d} p_i f(p_i) \geqslant f\left(\sum_{i=1}^{d} p_i^2\right). \tag{25}$$

Let $p = \sum_{i=1}^{d} p_i^2$. Then we have $1/d \leqslant p \leqslant 1$. Now,

$$f(p) = f\left(\sum_{i=1}^{d} p_i^2\right) = \log_2\left(\frac{1}{\sqrt{p}}\right) + p - 1. \tag{26}$$

It is easy to validate that $f(p)$ is monotonously increasing in $[1/2, 1]$ and monotonously decreasing in $[1/d, 1/2]$. So,

$$f(p) \geqslant f(1/2) = 0. \tag{27}$$

Equations (25) and (27) suggest that

$$\sum_{i=1}^{d} p_i f(p_i) \geqslant 0, \tag{28}$$

i.e.,

$$\sum_i -p_i \log_2(p_i) \geqslant \sum_i 2p_i(1-p_i). \tag{29}$$

That is to say,

$$\mathrm{EMM}(|\phi\rangle) \leqslant E_{VN}(|\phi\rangle). \tag{30}$$

□

Theorem 2 can be generalized to mixed states easily, which is stated by the following corollary.

Corollary 2. *For a qudit mixed state ρ, $EMM(\rho) \leqslant E_F(\rho)$, where $EMM(\rho)$ and $E_F(\rho)$ represent the entanglement measure based on minors and the entanglement of formation of ρ, respectively.*

It was shown [7] that, for a pure state, the entanglement of formation is equal to its entanglement cost. Moreover, if the hypothesis that the entanglement of formation is an additive quantity is proven to be true, then the entanglement of the formation of a mixed state is also equal to its entanglement cost [1, 43]. Theorem 2 and corollary 2 show that the EMM is bounded up by the entanglement of formation. This makes the EMM have an operational interpretation, i.e., providing a lower bound of the entanglement cost.

Corollary 3. *Let $E_N(\rho) = \log_2^{\|\rho^{\Gamma_A}\|_1}$ represent the logarithmic negativity of the density operator ρ. Hereafter, ρ^{Γ_A} represents the partial transpose of ρ over the subsystem A and $\|\rho\|_1$ represents the trace norm of ρ. Since $E_F(\rho) \leqslant E_N(\rho)$ [15], we can obtain the relation between $EMM(\rho)$ and $E_N(\rho)$ immediately as follows:*

$$EMM(\rho) \leqslant E_N(\rho). \tag{31}$$

Theorem 3. *For a qudit pure state $\rho = |\phi\rangle\langle\phi|$, $EMM(\rho) \leqslant N(\rho)$, where $N(\rho) = \|\rho^{\Gamma_A}\|_1 - 1$ represents the negativity of ρ [14].*

Proof. Let $|\phi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}|\mu_i\rangle|\nu_i\rangle$ be the Schmidt decomposition of the state $|\phi\rangle$. Then, by lemma 4 and equation (3), we have

$$\text{EMM}(\rho) = 4 \text{ Minors}(\Phi) = 2\sum_{i=1}^{d} \lambda_i(1-\lambda_i). \tag{32}$$

On the other hand, we have [15]

$$N(\rho) = \left(\sum_{i=1}^{d} \sqrt{\lambda_i}\right)^2 - 1 = 2\sum_{i<j} \sqrt{\lambda_i}\sqrt{\lambda_j}. \tag{33}$$

So,

$$\text{EMM}(\rho) - N(\rho) = 2\left(\sum_{i=1}^{d} \lambda_i(1-\lambda_i) - \sum_{i<j} \sqrt{\lambda_i}\sqrt{\lambda_j}\right) \tag{34}$$

$$= 2\left(1 - \sum_{i=1}^{d} \lambda_i^2 - \sum_{i<j} \sqrt{\lambda_i}\sqrt{\lambda_j}\right) \tag{35}$$

$$= 2\left(1 - \left(\sum_{i=1}^{d} \lambda_i\right)^2 + 2\left(\sum_{i<j} \lambda_i\lambda_j\right) - \sum_{i<j} \sqrt{\lambda_i}\sqrt{\lambda_j}\right) \tag{36}$$

$$= 2\left(\sum_{i<j} \sqrt{\lambda_i}\sqrt{\lambda_j}(2\sqrt{\lambda_i}\sqrt{\lambda_j} - 1)\right) \tag{37}$$

$$\leqslant 0. \tag{38}$$

Here we have used $\sum_{i=1}^{d} \lambda_i = 1$ in equation (36) and $2\sqrt{\lambda_i}\sqrt{\lambda_j} \leqslant \lambda_i + \lambda_j \leqslant 1$ in equation (37). The equality in equation (38) is true if and only if there exist only two nonzero Schmidt coefficients λ_i and λ_j such that $\lambda_i = \lambda_j$. Thus, when ρ is a pure state, we have

$$\text{EMM}(\rho) \leqslant N(\rho). \tag{39}$$

However, when ρ is a mixed state, the convexity of $N(\rho)$ [20], i.e., $N\left(\sum_i p_i\rho_i\right) \leqslant \sum_i p_i N(\rho_i)$ implies that equation (39) may not be true. □

4. Geometry interpretation

Exploring a geometric approach to quantify the measure of entanglement was first introduced by Shimony [44] in the setting of bipartite pure states, and then generalized to the multipartite setting (via projection operations of various ranks) by Barnum and Linden [45]. For a pure state $|\psi\rangle$, the geometric measure of entanglement, for short GME, is defined as

$$E_{\sin^2}(|\psi\rangle) = 1 - \Lambda_{\max}^2 = 1 - \max_{|\phi\rangle\in S_P} \|\langle\phi|\psi\rangle\|^2, \tag{40}$$

where S_P represents the set of separable pure states. For a mixed state ρ, it is defined as

$$E_{\sin^2}(\rho) = \inf \sum_{p_i,|\psi_i\rangle} p_i E_{\sin^2}(|\psi_i\rangle), \tag{41}$$

where $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$.

The Bures metric introduced by Vedral and Plenio [11] is indeed another form of the geometry measure of entanglement. For a mixed state σ, its Bures metric of entanglement is defined as

$$E_B(\rho) = \mathrm{Min}_{\sigma\in S} D_B(\sigma\|\rho) = 2 - 2\sqrt{\mathrm{Max}_{\sigma\in S} F(\sigma,\rho)}, \tag{42}$$

where $F(\sigma,\rho) = [\mathrm{tr}\{\sqrt{\rho}\sigma\sqrt{\rho}\}^{1/2}]^2$ is the so-called fidelity (or Uhlmanns transition probability) [46], and S represents the set of separable states (including pure states and mixed states).

Recently, Cao and Wang [47] presented a revised geometric measure of entanglement (RGME), which is defined as

$$\widetilde{E}_{\sin^2}(\rho) = 1 - \mathrm{Max}_{\sigma\in S} F(\rho,\sigma) = \mathrm{Min}_{\sigma\in S}(1 - F(\rho,\sigma)), \tag{43}$$

where $F(\sigma,\rho) = [\mathrm{tr}\{\sqrt{\rho}\sigma\sqrt{\rho}\}^{1/2}]^2$, and S denotes the set of separable states.

We can revise Cao's RGME by the method of convex roof as follows. For a pure state $|\psi\rangle$,

$$\approx{E}_{\sin^2}(|\psi\rangle) = 1 - \mathrm{Max}_{\sigma\in S} F(|\psi\rangle\langle\psi|,\sigma) = \mathrm{Min}_{\sigma\in S}(1 - F(|\psi\rangle\langle\psi|,\sigma)), \tag{44}$$

and for a mixed state ρ,

$$\approx{E}_{\sin^2}(\rho) = \inf \sum_{p_i,|\psi_i\rangle} p_i \approx{E}_{\sin^2}(|\psi_i\rangle), \qquad \rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|. \tag{45}$$

Using the method presented by Vedral [11], we can show that, for the two-qudit pure state $|\psi\rangle = \sum_{j=1}^{d}\sqrt{\lambda_j}|a_j\rangle|b_j\rangle$, the closest separable state to it by the Bures metric is $\sigma^* = \sum_{j=1}^{d}\lambda_j|a_j\rangle|b_j\rangle\langle a_j|\langle b_j|$. That is to say,

$$\mathrm{Max}_{\sigma\in S} F(\sigma, |\psi\rangle\langle\psi|) = F(\sigma^*, |\psi\rangle\langle\psi|) = \sum_{j=1}^{d}\lambda_j^2. \tag{46}$$

Therefore,

$$\approx{E}_{\sin^2}(|\psi\rangle) = 1 - \sum_{j=1}^{d}\lambda_j^2. \tag{47}$$

By lemma 4, we know

$$\mathrm{EMM}(|\psi\rangle) = 4\,\mathrm{Minors}(\Psi) = 2\left(1 - \sum_{j=1}^{d}\lambda_j^2\right) = 2\approx{E}_{\sin^2}(|\psi\rangle\langle\psi|). \tag{48}$$

This shows that, for a pure state, its entanglement measure EMM is exactly two times of its geometric entanglement measure $\approx{E}_{\sin^2}$. For a mixed state ρ, since both EMM(ρ) and $\approx{E}_{\sin^2}(\rho)$ are constructed by the convex roof, the relation for pure states keeps true.

5. Conclusion

We have constructed a new entanglement measure called EMM (the entanglement measure based on minors) by the convex roof method, which has been used to construct some important entanglement such as the entanglement of formation, concurrence, negativity and logarithmic negativity. The new entanglement measure has been defined as equation (3) for pure states, and equation (1) for mixed states. We have proved that EMM is a good entanglement measure since it satisfies the three basic postulates for an entanglement measure: vanishes on separable states, remains invariant under unitary transformation and decreases monotonously under

LOCC. We have also investigated the relations between EMM and some other important entanglement measures constructed by the axiomatic point of view, such as the entanglement of formation, negativity, logarithmic negativity and concurrence. It is discovered that EMM is always smaller than or equal to the entanglement of formation. This makes the EMM have an operational interpretation, i.e., providing a lower bound of the entanglement cost. The relations between EMM and another two important entanglement measurements, i.e., negativity and logarithmic negativity, have also been obtained (see corollary 3 and theorem 3). For a two-partite quantum system of arbitrary dimensions $|\phi\rangle$, by equation (2) and lemma 4, we have the relation $\text{EMM}(|\phi\rangle) = (C(|\phi\rangle)^2)$. That is to say, EMM is equivalent to concurrence. However, their definitions and methods of proof are completely different.

Lemma 4 shows that the computation of EMM can also be done by the eigenvalues of coefficient matrices like most of the other entanglement measures. However, the definition of EMM shows that its computation can be finished directly by the two-order minors of coefficients matrices. Therefore, compared with the other entanglement measures, one of the most important advantages for EMM is that it can be computed easily and quickly. Another advantage of EMM is that it can be interpreted by geometry (see equation (48)), i.e., $\text{EMM}(|\phi\rangle)$ is related to our modified Bures metric(see equations (44) and (45)) between $|\phi\rangle$ and the closest separable state. EMM can be generalized to multipartite quantum states easily. At present, by the numerical optimization procedure based on negativity and Von. Neumman entropy, etc, the five-qubit maximally entangled state and six-qubit maximally entangled state have been found by Brown *et al* [35] and Borras *et al* [36], respectively. It is believed that using the EMM will improve the efficiency of searching for maximally entangled multipartite state, and bring us hope to find maximally entangled states in the higher-dimensional Hilbert spaces.

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Appendix A

In this appendix, we will prove lemma 2.

Proof. By $(\overrightarrow{a}, \overrightarrow{b}) = 0$, we have

$$\sum_{i=1}^{n} a_i b_i^* = 0. \tag{A.1}$$

It follows that

$$\begin{aligned}\left|\sum_{i=1}^{n} a_i b_i^*\right|^2 &= \sum_{i=1}^{n} a_i b_i^* \sum_{j=1}^{n} a_j^* b_j \\ &= \sum_{i,j} a_i a_j^* b_j b_i^* \\ &= 0.\end{aligned} \tag{A.2}$$

By the same reason, we have

$$\sum_{i,j} a_i^* a_j b_j^* b_i = 0. \tag{A.3}$$

Therefore,

$$\sum_{i<j}|a_ib_j-a_jb_i|^2=\frac{1}{2}\sum_{i,j}|a_ib_j-a_jb_i|^2$$
$$=\frac{1}{2}\left(\sum_{i,j}|a_i|^2|b_j|^2+\sum_{i,j}|a_j|^2|b_i|^2-\sum_{i,j}a_ia_j^*b_jb_i^*-\sum_{i,j}a_i^*a_jb_j^*b_i\right)$$
$$=1. \tag{A.4}$$

Here, we have used the known conditions, $\sum_{i=}^{n}|a_i|^2=1$ and $\sum_{j=1}^{n}|b_j|^2=1$. □

Appendix B

In this appendix, we will prove lemma 3.

Proof. (1) Since $B=UA$, we have

$$b_{ij}=\sum_k u_{ik}a_{kj}. \tag{B.1}$$

Now

$$B(mr,ns)=b_{mr}b_{ns}-b_{ms}b_{nr} \tag{B.2}$$
$$=\sum_k u_{mk}a_{kr}\sum_j u_{nj}a_{js}-\sum_k u_{mk}a_{ks}\sum_j u_{nj}a_{jr} \tag{B.3}$$
$$=\sum_{k<j}(u_{mk}u_{nj}-u_{mj}u_{nk})(a_{kr}a_{js}-a_{ks}a_{jr}) \tag{B.4}$$
$$=\sum_{k<j}U(mk,nj)A(kr,js). \tag{B.5}$$

So,

$$|B(mr,ns)|^2=B(mr,ns)B(mr,ns)^*$$
$$=\left(\sum_{k<j}(u_{mk}u_{nj}-u_{mj}u_{nk})(a_{kr}a_{js}-a_{ks}a_{jr})\right)$$
$$\times\left(\sum_{k<j}(u_{mk}u_{nj}-u_{mj}u_{nk})(a_{kr}a_{js}-a_{ks}a_{jr})\right)^*$$
$$=\left(\sum_{k<j}U(mk,nj)A(kr,js)\right)\left(\sum_{k'<j'}U(mk',nj')A(k'r,j's)\right)^*. \tag{B.6}$$

Denote $\sum_{m<n,r<s}|B(mr,ns)|^2$ by M. Then

$$M=\sum_{m<n}\sum_{r<s}\left(\sum_{k<j}U(mk,nj)A(kr,js)\right)\times\left(\sum_{k'<j'}U(mk',nj')A(k'r,j's)\right)^*$$
$$=\sum_{m<n}\sum_{r<s}\left(\sum_{(k<j,k'<j')}U(mk,nj)U^*(mk',nj')A(kr,js)A^*(k'r,j's)\right)$$
$$=\sum_{(k<j,k'<j')}\left(\sum_{m<n}U(mk,nj)U^*(mk',nj')\right)\left(\sum_{r<s}A(kr,js)A^*(k'r,j's)\right)$$

$$= \sum_{k<j}\left(\sum_{m<n}|U(mk,nj)|^2\right)\left(\sum_{r<s}|A(kr,js)|^2\right)$$
$$+ \sum_{(k<j,k'<j')}^{(k\neq k' or j\neq j')}\sum_{m<n}U(mk,nj)U^*(mk',nj')\sum_{r<s}A(kr,js)A^*(k'r,j's). \quad \text{(B.7)}$$

By lemma 2, we know

$$\sum_{m<n}|U(mk,nj)|^2 = 1. \quad \text{(B.8)}$$

So,

$$\sum_{k<j}\left(\sum_{m<n}|U(mk,nj)|^2\right)\left(\sum_{r<s}|A(kr,js)|^2\right) = \sum_{k<j}\sum_{r<s}|A(kr,js)|^2. \quad \text{(B.9)}$$

On the other hand, when $k < j$, $k' < j'$ and $k \neq k'$ or $j \neq j'$, denote $\sum_{r<s} A(kr,js)A^*(k'r,j's)$ by N. Then,

$$N = \sum_{r<s}(a_{kr}a_{js} - a_{ks}a_{jr})(a^*_{k'r}a^*_{j's} - a^*_{k's}a_{j'r}*)$$
$$= \sum_{r<s}a_{kr}a_{js}(a^*_{k'r}a^*_{j's} - a^*_{k's}a_{j'r}*) - \sum_{r<s}a_{ks}a_{jr}(a^*_{k'r}a^*_{j's} - a^*_{k's}a_{j'r}*)$$
$$= \sum_{r<s}a_{kr}a_{js}(a^*_{k'r}a^*_{j's} - a^*_{k's}a_{j'r}*) + \sum_{r>s}a_{kr}a_{js}(a^*_{k'r}a^*_{j's} - a^*_{k's}a_{j'r}*)$$
$$= \sum_{r,s}a_{kr}a_{js}(a^*_{k'r}a^*_{j's} - a^*_{k's}a_{j'r}*)$$
$$= \sum_{r}a_{kr}a^*_{k'r}\sum_{s}a_{js}a^*_{j's} - \sum_{r}a_{kr}a^*_{j'r}\sum_{s}a_{js}a^*_{k's}. \quad \text{(B.10)}$$

The known conditions, $k < j$ and $k' < j'$, show that $k \neq j'$ or $j \neq k'$. So, by the orthogonality of A, we have $\sum_r a_{kr}a^*_{j'r}\sum_s a_{js}a^*_{k's} = 0$ and $\sum_r a_{kr}a^*_{k'r}\sum_s a_{js}a^*_{j's} = 0$. Thus, when $k < j$, $k' < j'$ and $k \neq k'$ or $j \neq j'$, we have

$$\sum_{r<s}A(kr,js)A^*(k'r,j's) = 0. \quad \text{(B.11)}$$

Substituting equations (B.11) and (B.9) into equation (B.7), we get

$$\sum_{m<n}\sum_{r<s}|B(mr,ns)|^2 = \sum_{k<j}\sum_{r<s}|A(kr,js)|^2. \quad \text{(B.12)}$$

This completes the proof of the first part.

(2) $C = AV$ implies that $C^T = V^TA^T$, where C^T represents the transpose of C. So, by the first part of this lemma, we have

$$\sum_{m<n}\sum_{r<s}|C^T(mr,ns)|^2 = \sum_{k<j}\sum_{r<s}|A^T(kr,js)|^2. \quad \text{(B.13)}$$

It is easy to check that for any matrix A, we have

$$\sum_{k<j}\sum_{r<s}|A^T(kr,js)|^2 = \sum_{k<j}\sum_{r<s}|A(kr,js)|^2. \quad \text{(B.14)}$$

It follows that

$$\sum_{m<n}\sum_{r<s}|C(mr,ns)|^2 = \sum_{k<j}\sum_{r<s}|A(kr,js)|^2. \quad \text{(B.15)}$$

This completes the proof of the second part. □

Appendix C

In this appendix, we will prove lemma 4.

Proof. Suppose that the coefficient matrix corresponding to $|\phi\rangle$ in basis $\{|i, j\rangle \mid 1 \leqslant i \leqslant d_1, 1 \leqslant j \leqslant d_2\}$ is $\Phi = (a_{i,j})_{d_1 \times d_2}$. By the Scmidt decomposition theorem, we know that $\sqrt{\lambda_i}$'s $(i = 1, \ldots, d)$ are exactly the singular values of Φ, i.e., the nonnegative square roots of the eigenvalues of $AA^\dagger$. Denote the $d_1 \times d_2$ matrix,

$$\begin{pmatrix} \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_d}) & 0 \\ 0 & 0 \end{pmatrix},$$

by Σ. Then, by the singular-value decomposition theorem there exist unitary matrices $U_{d1 \times d_1}$ and $V_{d_2 \times d_2}$ such that

$$\Phi = U \Sigma V. \tag{C.1}$$

By lemma 3, we have

$$\mathrm{Minors}(\Phi) = \mathrm{Minors}(\Sigma). \tag{C.2}$$

By corollary 1, we can get

$$\mathrm{Minors}(\Sigma) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2} \sum_{i \neq j} \lambda_i \lambda_j = \frac{1}{2} \sum_i \lambda_i (1 - \lambda_i). \tag{C.3}$$

Therefore,

$$\mathrm{Minors}(\Phi) = \frac{1}{2} \sum_i \lambda_i (1 - \lambda_i). \tag{C.4}$$

This completes the proof. □

Appendix D

In this appendix, we will prove lemma 5.

Proof.

$$\begin{aligned} f(\overrightarrow{\lambda}) &= \frac{1}{2} \sum_{i=1}^{d} \lambda_i - \frac{1}{2} \sum_{i=1}^{d} \lambda_i^2 \\ &= \frac{1}{2} - \frac{1}{2} \sum_{i=1}^{d} \lambda_i^2 \\ &\leqslant \frac{1}{2} - \frac{1}{2} \left(\frac{\sum_{i=1}^{d} \lambda_i}{d} \right) \\ &= \frac{1}{2} \left(1 - \frac{1}{d} \right), \end{aligned}$$

where the equality is true if and only if $\forall\, i, \lambda_i = \frac{1}{d}$. □

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